LAMINAR AND TRANSITIONAL BOUNDARY LAYER ENTROPY GENERATION OVER A FLAT PLATE UNDER FAVORABLE AND ADVERSE PRESSURE GRADIENTS

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ABSTRACT
Minimizing entropy generation is important to improve the efficiency of any system. The objective of this study is to use computational fluid dynamics (CFD) to elucidate the effects of pressure gradients on entropy generation rates in laminar and transitional boundary layers. The commercial CFD software, ANSYS FLUENT, is employed. The favorable and adverse pressure gradients are generated using various curved slip top walls. Bypass transition is simulated using the mean inlet velocity and Reynolds stresses from the direct numerical simulation (DNS) conducted by Nolan and Zaki [1]. Various turbulence and transitional models are employed and the results are compared to the DNS data. The factor of safety method is used to evaluate numerical error and grid uncertainties. Three systematically refined meshes are used to evaluate grid convergence. Monotonic convergence is achieved for all simulations with small grid uncertainties. The boundary layer correlation function, $F(\lambda)$, the shear stress correlation, $S(\lambda)$, and the dissipation coefficient, $C_d$, are calculated for the laminar CFD results. The dissipation coefficient, $C_d$ and the intermittency, $\gamma$, are also calculated for the bypass transition CFD results. The laminar CFD results show better agreement with the correlation developed by McEligot than with the Thwaites correlation for $F(\lambda)$ and $S(\lambda)$. Overall, the percentage differences between the CFD results and the correlations increase as the magnitude of the pressure gradient variable, $\beta$, increases. The solvers and turbulence models in the transitional simulations are similar to the study by Ghasemi et al [2]. However, this study uses a much finer grid and improved boundary conditions for the inlet. These changes show an improvement for most turbulence models by comparison with the DNS results, especially for the location of transition.

INTRODUCTION
Entropy is the irreversible energy loss in fluid flow. Entropy generation is proportional to the efficiency of a system; therefore, minimizing entropy generation is necessary to improve the efficiency of any system. Minimizing entropy generation is necessary for cooling electronic devices or nuclear reactors, thermal heat exchangers, and much more. Entropy generation cannot be directly measured as it is a non-conserved variable, which means that entropy generation analysis requires the usage of the first and second law of thermodynamics. Nonetheless, entropy generation can be approximated using four mechanisms: dissipation in the mean and the fluctuating velocity field and heat flux in the mean and fluctuating temperature field. Different studies focus on determining which mechanisms are dominant for specific flows.

Experimental studies are paramount to study entropy generation, which is typically presented in the form of empirical correlations. Sampling transient characteristics of transitional or turbulent flow in experimental studies makes entropy generation approximation difficult. Data can be nonconditionally sampled, laminar-conditioned, turbulent-conditioned, and intermittency weighted (which is a combination of the two conditionally sampled methods). Experimental results analyzed by Walsh et al. [3] compare the
boundary layer flow with zero and small favorable pressure gradients. Adeyinka and Naterer [5] post-processed particle image velocimetry (PIV) data of turbulent flows within a channel. The experiment involved flows with Reynolds numbers based on friction velocity, $Re_\theta$, from 187-399. The results demonstrated a 3% deviation from the results of White [6]. Comparisons with direct numerical simulations (DNS) confirm that turbulent entropy production is modeled correctly. Approximate entropy generation using PIV experimental methods has showed an uncertainty of 9.34% for free convection flow in an enclosure [7] and 11.67% for turbulent flows.

DNS is a proven tool in elucidating flow physics. DNS completely resolves all the laminar and turbulent length scales and thus can be used as a numerical benchmark to evaluate simulations using various turbulence models. Since two-thirds of the entropy generation occurs in the viscous layer of turbulent flows (defined as $y^+ \approx 30$), DNS studies on entropy generation tend to focus on this region. McEligot et al. [8] analyzed results from direct numerical simulations of turbulent boundary layer flow with zero and small favorable pressure gradients with $Re_\theta = 300$ to 1410 [9, 10]. The study demonstrated that dissipation is nearly universal within the viscous layer of turbulent boundary layer flows with zero and small favorable pressure gradients. The study showed that existing methods for approximating $S^\theta$, developed by Rotta [11], are inaccurate for the given flow characteristics. McEligot et al. [12] similarly analyzed results from a DNS [13] of turbulent channel flow with zero and small favorable pressure gradients with $Re_\theta = 295, 754,$ and 1283. The study compared two methods for approximating entropy generation. The first method evaluates the fluctuating gradients forming the dissipation term in the turbulent enthalpy equation and the second method evaluates an approximate analogy to laminar flow employing assumed boundary layer (and other) approximations. The study demonstrated that while both methods predict similar $S^\theta$ values, the entropy production is under-predicted in the “linear” layer and over-predicted in the rest of the viscous layer from the second method. Another study by McEligot et al. [14] compared the entropy generation in a DNS of turbulent boundary layer and channel flows [13, 15]. The study demonstrated that for significantly large favorable pressure gradients, the point-wise entropy at the boundary of the viscous layer is relatively insensitive for both flow types but the integral over the area of the viscous layer decreases moderately only for boundary layer flows. Similarly, Walsh and McEligot [16] improved an existing correlation for the dissipation coefficient, $C_{\omega}$, using data from multiple DNS’s of turbulent boundary layer and channel flows with zero and mild favorable pressure gradients and low $Re_\theta$ [10, 13, 17, 18]. Walsh et al. [19] analyzed a DNS of by-pass transition boundary layer flow for $Re_\theta = 115$ to 520 [20, 21]. The study demonstrated that the term for turbulent convection in the turbulent kinetic energy balance is significant for the transition region resulting in more turbulent energy being produced than is dissipated. The study showed that a popular approximation method currently in over estimates the dissipation coefficient by as much as 17%, therefore the approach developed by Rotta [11] is more accurate for transitional boundary layers.

More recently, a CFD study by Ghasemi et al. [2] was conducted that evaluated the accuracy of different turbulence models for predicting skin friction coefficient, momentum thickness, and entropy generation for bypass transition. The models implemented in the study were the $k-e$ model, SST $k-\omega$ model, $k-\omega$ 4 equation model, $k-kl-\omega$ 3 equation model, and the Reynolds stress model. The mesh used in this study contained 149,089 grid points. The inlet turbulence intensity was a constant 3% with a turbulent length scale equal to the boundary layer thickness. The study showed that the all the RANS models predict the onset of transition much earlier than the DNS [1].

**OBJECTIVE AND APPROACH**

The objective of this study is to elucidate the entropy generation in steady, unheated, incompressible, two-dimensional laminar boundary layers over a flat plate with and without pressure gradients and transitional boundary layers using CFD. The laminar boundary layer simulations with and without pressure gradients are compared against the Falkner-Skan and Blasius solutions, respectively. The CFD results evaluate the accuracy of the correlations for boundary layer parameters developed by Thwaites [6, 22] and McEligot and Walsh [23].

The bypass-transitional flow simulations focus on evaluating the flow behavior and capability of various turbulence models on predicting boundary layer entropy generation. These characteristics are compared to the DNS results from Nolan and Zaki and a recent CFD study by Ghasemi et al.

The commercial CFD software ANSYS FLUENT 14.0 [24] is used to conduct all the simulations. The Falkner-Skan equations are used to design seven different pressure gradients including zero pressure gradient flow. Quantitative solution verification is conducted using three systematically refined structured grids with the finest grid up to 1 million grid points. The numerical errors and associated uncertainties are evaluated using the factor of safety method by Xing and Stern [25].

**GOVERNING EQUATIONS**

FLUENT uses the mass conservation equation [24],

$$ \nabla \cdot (\rho \mathbf{V}) = 0 \quad (1) $$

and the momentum conservation equation [24],

$$ \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = -\nabla p + \nabla \cdot (\mathbf{f}) \quad (2) $$
The coupled pressure-velocity coupling scheme is used. For spatial discretization, second order scheme is used for pressure and momentum.

**SIMULATIONS**

The inlet boundary condition for the laminar simulations is set to pressure inlet with the total pressure specified. The laminar outlet boundary condition is set to pressure outlet with the gage pressure set to zero. The bottom boundary condition is set to a no-slip wall. The plate has a length of 28 m. The top wall boundary condition is a zero shear stress wall or slip wall and is shaped to create the desired pressure gradients using the area calculated from the Falkner-Skan’s equation plus the displacement thickness calculated from the Falkner-Skan solutions to accommodate the boundary layer growth.

The boundary conditions and geometry in the bypass transitional simulations differ from those in the laminar simulations to match the DNS conditions. The bypass transitional inlet boundary condition is a velocity inlet with a mean velocity profile based on the DNS velocity inlet. This velocity profile is based on the Blasius velocity profile at $Re_\delta = U_\delta \delta_0/\nu = 800$ based on the inflow free-stream velocity, the 99% boundary layer thickness, $\delta_0$, and the kinematic viscosity. The turbulence at the inlet is determined using the Reynolds stresses at the inlet of the DNS. This is described in more detail in later sections. The outlet boundary condition is set to outflow. No-slip wall boundary condition is set for the plate. The length of the plate, $L_x$, is the same as the DNS study, i.e. $L_x/\delta_0 = 600$. The top wall boundary condition is set to a slip wall and the shape matches the geometry provided by Nolan and Zaki from the DNS. Their wall shape is based on the displacement thickness, just as the laminar simulations were set up.

Seven values for $\beta$ were simulated with three values for adverse pressure gradients, three for favorable pressure gradients, and one for a zero pressure gradient. A grid convergence study was run for all $\beta$ values with three systematically refined grids using a grid refinement ratio of 2 for the laminar simulations and $(2^{1/2})$ for the transitional simulations. The finest grid for the laminar simulations contains $1 \times 10^6$ grid points with 1,227 in the streamwise direction and 827 in the plate normal direction. The medium and coarse grids have $2.1 \times 10^5$ and $5.2 \times 10^4$, respectively. The bypass transitional simulations have fine, medium, and coarse grids containing $1 \times 10^5$, $5 \times 10^4$, and $1.25 \times 10^4$, respectively. The turbulence models in the bypass transition simulations are the $k-\epsilon$ model, $k-\omega$ SST model, transitional $k-\omega$ 4 equation model, and the Reynolds stress model (RSM).

All simulations are conducted on a Dell Optiplex 990 computer with an Intel® Core™ i7-2600 CPU and 8 GB’s of RAM. The simulations took an average of 8 hours to converge. Results are post-processed using Tecplot 360.

**Laminar Boundary Conditions**

For each simulation, the parameter $\beta$ is set to a constant value, where $\beta$ represents the streamwise pressure gradient, $dp/dx$. When $\beta < 0$, the pressure gradient is negative or adverse and while $\beta > 0$, the pressure gradient is positive or favorable. A zero pressure gradient occurs when $\beta = 0$. The variable $\beta$ relates to the Falkner-Skan power-law parameter, $m$ [26, 27],

$$
\beta = \frac{2m}{1+m}
$$

(3)

one can write the Falkner-Skan equation presented in [6, 26, 27],

$$
U_a(x) = Kx^m
$$

(4)

where $U_a(x)$ is the freestream velocity as it changes with the streamwise location, $x$, and $K$ is a constant that sets the inlet velocity to remain at 1 m/s for these computations.

Since the Falkner-Skan equations have a fictitious origin, a pressure inlet boundary is used. The static gage pressure is determined for each $x$ location using the Bernoulli’s equation. Bernoulli’s equation is,

$$
\frac{dp}{dx} = \rho U_a(x) \frac{dU_a(x)}{dx}
$$

(5)

where $dp/dx$ is the change in pressure over $x$ and $\rho$ is the density of the fluid. When (4) is inserted into (5) and solved using integration the equation becomes,

$$
p_i = p_e + \frac{1}{2} \rho \left( Kx^m \right)^2
$$

(6)

where $p_i$ is the total gage pressure chosen at the lowest value of $U_a(x)$ which is at the inlet for favorable pressure gradient and at the outlet for adverse pressure gradient. The free stream static gage pressure is $p_e$, which is zero at the outlet.

In order to determine the change in the shape of the wall to create the appropriate pressure gradient, the flow conservation equation is,

$$
Q = A(x)U_a(x)
$$

(7)

where $Q$ is the fixed volumetric flow rate with units m$^3$/s, and $A(x)$ is the cross sectional area with units m$^2$. Since the simulations are two-dimensional, the following is true,

$$
A(x) = h(x)
$$

(8)

where $h(x)$ is the height in the plate normal direction. The wall is additionally expanded by the value of the displacement boundary layer thickness. The dimensionless form for the displacement thickness is determined using [27, 28],

$$
\eta^* = \lim_{\eta \to \infty} (\eta - f) 
$$

(9)

which is converted into dimensional form with:

$$
\delta^* = \frac{\eta^*}{U_a(1+m)} \sqrt{\frac{2\nu x}{U_a}}
$$

(10)

Adding the displacement thickness to the top wall is necessary to ensure the flow rate remains constant despite the increasing boundary layer thickness. So the final equation for the wall height $h(x)$ after combining (7), (8),(10):
\[ h(x) = \frac{O}{U_h(x)} + \delta^* \]  

(11)

Figure 1 shows a demonstration of the geometry and the boundary conditions used for each boundary. Due to the fictitious origin of the Falkner-Skan equations, the inlet is set at \( x = 2 \) to shape the wall.

**FIGURE 1: REPRESENTATION OF ADVERSE PRESSURE GRADIENT GEOMETRY**

**Boundary Conditions for Bypass Transition**

The spanwise-averaged and time-average mean velocity values and Reynolds stresses from the DNS data are prescribed at the inlet. The turbulent kinetic energy, \( k \), is shown in the Nomenclature section. For the \( k-\varepsilon \) model, \( \varepsilon \) is estimated at the inlet as,

\[ \varepsilon = 0.09^{3/4} \frac{k^{3/2}}{0.45 \delta_0} \]  

(12)

and for the \( k-\omega \) model, \( \omega \) is estimated at the inlet as,

\[ \omega = 0.09^{-3/4} \sqrt{k} \]  

(13)

Formulas (12) and (13) come from the FLUENT User’s Manual. The Reynolds stresses are specified directly at the inlet for the RSM.

**ANALYTICAL CORRELATIONS AND CFD POST-PROCESSING**

**Variable Calculations**

The freestream velocity, \( U_{in} \), is defined as,

\[ U_{in} = 0.99 \max(\alpha \{ y = 0 : y_{max} \}) \]  

(14)

which is the 99% of the maximum streamwise velocity for a velocity profile at a specific \( x \) location. The boundary layer thickness, \( \delta \), is the \( y \) location when \( u \geq U_{in} \). The displacement thickness equation is defined as,

\[ \delta^* = \int_0^\infty 1 - \frac{u}{U_{in}} dy \]  

(15)

The momentum thickness, \( \theta \), is analytically calculated with,

\[ \theta = \frac{\mu v x}{\sqrt{((b-1)m+1)Kx^w}} \]  

(16)

\[ \theta = \int_0^\infty \frac{u}{U_{in}} \left( 1 - \frac{u}{U_{in}} \right) dy \]  

(17)

both (15) and (17) are integrated to the value of \( \delta \) that is discussed earlier to avoid the indefinite integral.

Comparisons are made to a modified version of Thwaites’ correlation method [6, 22]. The two correlations calculate the boundary layer correlation function, \( F(\lambda) \), as,

\[ F(\lambda) = a - b\lambda \]  

(18)

while \( F(\lambda) \) is computed from the CFD results as,

\[ F(\lambda) = 2(S(\lambda) - \lambda(2 + H(\lambda))) \]  

(19)

where the shape function, \( H(\lambda) \), is define as,

\[ H(\lambda) = \frac{\delta^*}{\theta} \]  

(20)

The wall shear stress correlation is calculated as,

\[ S(\lambda) = \frac{C_s(\lambda - \lambda_0)\varepsilon_s}{2} \]  

(21)

while \( S(\lambda) \) is calculated from the CFD results and compared to (21) with,

\[ S(\lambda) = \frac{r_s\theta}{\mu U_{in}(x)} \]  

(22)

Both approaches use the same equations but use different values for the constants in the equations as shown in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Thwaites [6, 22]</th>
<th>McEligot and Walsh [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.45</td>
<td>0.44105</td>
</tr>
<tr>
<td>( b )</td>
<td>6</td>
<td>5.30934</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>-0.09</td>
<td>-0.068</td>
</tr>
<tr>
<td>( C_s )</td>
<td>2</td>
<td>2.1348</td>
</tr>
<tr>
<td>( E_s )</td>
<td>0.62</td>
<td>0.58706</td>
</tr>
</tbody>
</table>

For the laminar flow, only the vicious dissipation for the mean velocity profile contribute to entropy generation. The point-wise entropy generation rate equation applies for steady, pure laminar two-dimensional boundary layer flows without significant fluctuations [19]:

\[ TS^* \{ y \} \approx \mu \left( \frac{\partial U}{\partial y} \right)^2 \]  

(23)

The wall shear stress is provided by FLUENT and is defined as [24]:

\[ \frac{O}{U_h(x)} + \delta^* \]
\[ \tau_w(x) = \mu \frac{\partial u}{\partial y} \bigg|_{x=0} \] (24)

The integral over the boundary layer of the point-wise entropy generation rate provides the entropy generation rate per unit area, \( S'' \) [19]:

\[ TS'' = \int_0^\theta S'' dy \] (25)

Fluctuations are present in bypass transitional flow therefore (23) and (25) do not apply to the bypass transition simulations. For these cases a different equation is used,

\[ \left( S''(\delta) \right)^+ \approx \int_0^\theta \left( \frac{\partial U'}{\partial y^+} \right)^2 dy^+ - \int_0^\delta \left[ (u'^2) - (v'^2) \right] \frac{\partial U'}{\partial x^+} dy^+ - \left( \frac{d}{dx^+} \right) \int_0^\delta U' \left( 1/2 \right) (q^2) dy^+ \] (26)

A dimensionless form of (25) is the energy dissipation coefficient [19],

\[ C_d = \frac{TS''}{\rho U'_n} = \left( S''(\delta) \right)^+ \left( \frac{C_f}{2} \right)^{1/2} \] (27)

The dissipation coefficient is multiplied by \( Re_0 \) and compared to,

\[ C_d Re_0 = 0.1740 + 0.3315 \lambda + 0.7881 \lambda^2 \] (28)

which comes from the McEligot and Walsh correlations.

The intermittency is calculated as,

\[ \gamma = \frac{(C_f - C_{lam})}{(C_{turb} - C_{lam})} \] (29)

The intermittency is compared to the transition length,

\[ \eta_l = \frac{(x - x_n)}{(x_e - x_n)} \] (30)

The intermittency in this paper is based on the skin friction coefficient. The x value for the beginning of transition, \( x_n \), is when the intermittency \( \gamma = 0.005 \) and the x value for the end of transition, \( x_e \), is when \( \gamma = 0.995 \).

The laminar skin friction coefficient predicted as,

\[ C_{lam} = 0.664 / Re_e^{0.5} \] (31)

and the turbulent skin friction coefficient predicted as,

\[ C_{turb} = 0.455 / \ln^2(0.06 Re_e) \] (32)

Calculating the Reynolds stresses in the flow domain is necessary to determine the entropy generation per unit area. For all models except the RSM (where the Reynolds stresses are calculated by the model) the following is using the Boussinesq Approach:

\[ \vec{u'} \vec{v'} = -v' \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) \]

\[ \vec{u'}^2 = \frac{2}{3} - 2v' \left( \frac{\partial u'}{\partial x} \right) \] (33)

\[ \vec{v'}^2 = \frac{2}{3} - 2v' \left( \frac{\partial v'}{\partial y} \right) \]

as outlined in the ANSYS FLUENT User’s Manual.

Solution Verification

Solution verification is important to estimate the numerical errors and grid uncertainties of a CFD simulation. The relative difference between CFD results \( \hat{S} \) and the correlation values \( A \) is calculated as:

\[ \delta_s = \left| \frac{A - \hat{S}}{A} \right| \times 100\% \] (34)

The grid verification study requires the use of the following equations from [29]:

\[ \varepsilon_{21} = \hat{S}_3 - \hat{S}_1 \] (35)

\[ \varepsilon_{32} = \hat{S}_2 - \hat{S}_1 \] (36)

\[ \langle R_G \rangle = \left\| E_{21} \right\| / \left\| E_{32} \right\| \] (37)

\[ \langle P_G \rangle = \ln \left( \left\| E_{32} \right\| / \left\| E_{21} \right\| \right) / \ln(r_{Gi}) \] (38)

here \( \hat{S}_1, \hat{S}_2, \hat{S}_3 \) represents the fine, medium, and coarse grid solutions, respectively. The grid refinement ratio is 2 for the laminar simulations and \( 2^{1/2} \) for the transitional simulations.

The factor of safety method by Xing and Stern is used [25, 30] to estimate the numerical errors and grid uncertainties. The ratio of the estimated order of accuracy and theoretical order of accuracy is defined as,

\[ P = \left\{ \frac{P_G}{p_{th}} \right\} \] (39)

where \( P_{th} \) is the theoretical order of accuracy of the numerical scheme that is set to 3 and 2 for the laminar and transitional simulations, respectively.

The grid uncertainty, \( U_{Gi} \) from [25], is defined as,
\[
\delta_{RE} = \frac{\epsilon_{21}}{P_{PE} - 1}
\]  
\[
U_G = \begin{cases} 
[1.6P + 2.45(1 - P)]\delta_{RE} & 0 < P \leq 1 \\
[1.6P + 14.8(P - 1)]\delta_{RE} & P > 1 
\end{cases}
\]  
which is a percentage of the correlation value at the same streamwise location.

**RESULTS AND DISCUSSION**

**Solution Verification**

The results presented are only from the fine grid simulations for all simulations, which are run to convergence for a convergence tolerance of \(1 \times 10^{-10}\) to ensure that the iterative errors are much smaller than the grid errors such that the former can be neglected. Dimensionless variables are presented so results are universally applicable.

Grid convergence studies are run for all values of \(\beta\). Ideal results for a grid convergence study is when monotonic convergence is achieved (if \(0 < R_G < 1\)), and the value of \(U_G(\%)\) minimized. Table 2 shows that all simulations achieve monotonic convergence. The grid uncertainty is below 1% for all cases except the \(F(\lambda)\) variable when \(\beta\) is 0.25. More importantly, the dissipation coefficient grid uncertainty remains below 0.005% for all beta values.

Table 3 shows that monotonic convergence is achieved for both variables. The grid uncertainty is below 1% for both variables used in the study. This shows that the results presented are independent of grid resolution.

**TABLE 3: TRANSITIONAL GRID CONVERGENCE STUDY RESULTS**

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(R_G)</th>
<th>(P)</th>
<th>(U_G(%))</th>
<th>(R_G)</th>
<th>(P)</th>
<th>(U_G(%))</th>
<th>(R_G)</th>
<th>(P)</th>
<th>(U_G(%))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.371</td>
<td>0.477</td>
<td>0.618</td>
<td>0.238</td>
<td>0.690</td>
<td>0.034</td>
<td>0.509</td>
<td>0.325</td>
<td>0.001</td>
</tr>
<tr>
<td>0.25</td>
<td>0.679</td>
<td>0.186</td>
<td>1.499</td>
<td>0.284</td>
<td>0.606</td>
<td>0.085</td>
<td>0.604</td>
<td>0.242</td>
<td>0.002</td>
</tr>
<tr>
<td>0.05</td>
<td>0.347</td>
<td>0.509</td>
<td>0.153</td>
<td>0.279</td>
<td>0.614</td>
<td>0.104</td>
<td>0.690</td>
<td>0.178</td>
<td>0.004</td>
</tr>
<tr>
<td>0</td>
<td>0.310</td>
<td>0.564</td>
<td>0.144</td>
<td>0.298</td>
<td>0.583</td>
<td>0.131</td>
<td>0.707</td>
<td>0.167</td>
<td>0.004</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.320</td>
<td>0.548</td>
<td>0.189</td>
<td>0.292</td>
<td>0.593</td>
<td>0.124</td>
<td>0.727</td>
<td>0.153</td>
<td>0.004</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.289</td>
<td>0.597</td>
<td>0.197</td>
<td>0.268</td>
<td>0.633</td>
<td>0.129</td>
<td>0.735</td>
<td>0.148</td>
<td>0.004</td>
</tr>
<tr>
<td>-0.14</td>
<td>0.368</td>
<td>0.481</td>
<td>0.327</td>
<td>0.456</td>
<td>0.378</td>
<td>0.670</td>
<td>0.708</td>
<td>0.166</td>
<td>0.004</td>
</tr>
</tbody>
</table>

**FIGURE 2: \(\beta\) VERSUS \(x/L\)**

**\(\beta\) Values**

Figure 2 compares the \(\beta\) values from the Falkner-Skan equations with the \(\beta\) values computed as,

\[
\beta = \frac{2 \ln \left( \frac{U_h(x)}{K} \right)}{\ln(x) + \ln \left( \frac{U_h(x)}{K} \right)}
\]

from the CFD results.

The inconsistencies between the CFD \(\beta\) values and the Falkner-Skan \(\beta\) values is due to the unphysical nature of the inlet boundary condition. Data is only collected from the \(x/L > 60\%\) region to avoid issues with the inlet boundary condition.

**Variables in Streamwise Direction**

Figure 3 to Figure 5 show the values for the variables \(F(\lambda), S(\lambda), \) and \(C_dR_0\) in the streamwise direction, respectively. The variables are constant for each constant value of \(\beta\) along the streamwise direction for both correlations. All three figures show that the more adverse the pressure gradient becomes, the less the computational results remain constant in
the streamwise direction. The figures show that as the magnitude of $\beta$ increases, the difference between the two correlations increases. This trend is more explicit for the adverse pressure gradients than for the favorable pressure gradients.
Use of the correlations requires $\beta$ stay within a set of bounds. This is because separation occurs as the pressure gradient becomes more adverse and boundary layer behavior in extremely favorable pressure gradients is not consistent with the Falkner-Skan predictions. These discrepancies could explain why the variance of the CFD results increases as the magnitude of $\beta$ increases.

**Effect of $\beta$ Values**

Figure 6 to Figure 8 show the values for $F(\lambda)$, $S(\lambda)$, and $C_d R_e \theta$ for different values of $\beta$, respectively. These values are all evaluated at the $x/L = 86\%$.
increases. The figures show that $\delta_{\text{th}}$ increases as the magnitude of $\beta$ increases. The percent difference between the Thwaites’ correlation and the CFD results exceeds 5% at $\beta$ values -0.14, -0.1, 0.25, and 0.5 for $F(\lambda)$ and $S(\lambda)$. The percent difference between McEligot’s correlation and the CFD results remains under 2.5% for all $\beta$ values except at the largest adverse pressure gradient.

**Bypass - Transition Flow**

The bypass transition simulation results are compared to the DNS and the CFD results by Ghasemi et al. [2]. The simulations by Ghasemi et al. differ from the current simulations based on the inlet conditions and meshes. The current simulation uses a finer mesh and prescribes the $k$, $\omega$, $\varepsilon$, or Reynew’s stress values at the inlet, depending on the model being used, to match the conditions used in the DNS simulation. Additionally, this study also examines entropy generation and compares CFD predictions with that post-processed from the DNS results.

**Variables**

shows how $Re_{\theta}$ varies with $Re_{e}^{1/2}$. The DNS data is very linear until $Re_{e}^{1/2} \approx 450$ where the slope gradually increases until $Re_{e}^{1/2} \approx 600$ after which the slope remains relatively constant. The $k-\omega$ 4 equation model results demonstrate a similar trend and closely resemble the DNS results but remain slightly lower throughout the domain. The slope of the $k-\varepsilon$ model remains constant throughout the entire domain, only showing similarity to the DNS results near the inlet. The RSM has a very minor slope change around $Re_{e}^{1/2} \approx 250$ but remains very similar to the $k-\varepsilon$ model results. The $k-\omega$ model shows closer relationship to the DNS data than the $k-\varepsilon$ and RSM’s but has a sudden increase in slope at $Re_{e}^{1/2} \approx 275$. All models from Ghasemi et al. diverge from the DNS data at a lower $Re_{e}^{1/2}$ value than the corresponding model in this study.

Figure 10 and Figure 11 demonstrate similarities between the models and the DNS data. The DNS data has a linear slope until $Re_{e}^{1/2} \approx 450$ where both $C_f$ and $C_d$ increase sharply and level off again to a linear slope. Similar to , the $k-\omega$ 4 equation model is closest to the DNS data except $C_f$ and $C_d$ are 4% lower than the DNS data at the onset of transition. The $k-\varepsilon$ and RSM’s become transitional at the inlet and remain turbulent throughout the flow field. The $k-\omega$ model shows closer agreement to the DNS for the first section of the flat plate and the last section but transition occurs at $Re_{e}^{1/2} \approx 200$, far before the $k-\omega$ 4 equation and the DNS data. All models from Ghasemi et al. diverge from the DNS data at a much lower $Re_{e}^{1/2}$ value than the corresponding model in this study.
Figure 12 shows that all turbulence models examined herein predict transition onset earlier than the DNS data. The k-ω 4 equation model is the closest in terms of predicting transition location but ends 5% above 1, the value the DNS data shows as fully turbulent. The k-ω, k-ε, and RSM’s demonstrate very similar trends and have steeper slopes than the other model and the DNS data except the k-ε model is the only one that increases above 1 and decreases back to 1.

CONCLUSIONS

This study uses CFD to elucidate the effects of pressure gradients on boundary variables $F(\lambda)$ and $S(\lambda)$ and entropy generation rates in bypass-transitional flows without a pressure gradient. Rigorous quantitative solution verification is conducted and monotonic convergence is achieved for all variables on three systematically refined grids, with the finest mesh containing up to $1 \times 10^6$ grid points. The results are compared to analytical correlations for the laminar boundary layer flows and to DNS data and a recent CFD study for bypass transitional flows.

The laminar CFD results demonstrate that entropy generation in the boundary layer increases as the pressure gradient becomes more favorable. The boundary layer growth across the plate increases as the pressure gradient becomes more adverse. The correlations predict constant values for the variables $F(\lambda)$, $S(\lambda)$, and $C_g Re_\theta$ in the streamwise direction, however the CFD results show an up to 1.5% variation in the variables along the flat plate under strong adverse pressure gradients. These variations could be the results of comparing the approximations of the Falkner-Skan solutions with the full solution of the Navier-Stokes solutions for the flow field. This study demonstrates that the correlation developed by McEligot more accurately predicts the $F(\lambda)$, $S(\lambda)$, and $C_g Re_\theta$ of laminar flow over a flat plate with streamwise pressure gradients. The predictions from McEligot’s correlation for $C_g Re_\theta$ are also very reasonable and can be implemented to other scenarios with little incident.

The bypass-transitional results show that the k-ω 4 equation model accurately predicts the boundary layer behavior and entropy generation for bypass-transitional flows. All models predict transition onset earlier than the DNS data except for the k-ω 4 equation model. This model shows a 4% difference in the onset of transition from the DNS data based on the skin friction coefficient. Overall the present study shows improvements over the CFD results by Ghasemi et al. likely due to the much finer grid used and the more accurate inlet boundary conditions for turbulent structures.

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 NOMENCLATURE

$\langle \_ \rangle$ = Profile averaged quantity
$\| \cdot \|_2$ = L2 norm
$| |$ = Absolute value
$A$ = Area
$a,b$ = Constants in Thwaites and McEligot correlation methods
$C_i$ = Correlation constant
$E_s$ = Correlation exponent
$F(\lambda)$ = Boundary layer correlation function
$f, f', f''$ = Falkner-Skan function, first and second derivative, respectively
$H(\lambda)$ = Shape factor correlation
$h$ = Height in the wall-normal direction
$K$ = Falkner-Skan constant
$k$ = Turbulent kinetic energy, $u'^2 + v'^2 + w'^2$
$m$ = Falkner-Skan power law parameter
$P$ = Distance metric to the asymptotic range
$p$ = Pressure
$\langle p_{\theta} \rangle$ = Profile averaged order of accuracy
$p_{th}$ = Theoretical order of accuracy
$Q$ = Volumetric flow rate
$q^2$ = Sum of velocity fluctuations squared, $u'^2 + v'^2 + w'^2$
$\langle R_{\theta} \rangle$ = Profile averaged convergence ratio
$G$ = Grid refinement ratio, $\Delta x_\theta / \Delta x_\theta$ and $\Delta x_\theta / \Delta x_\theta$
$\hat{S}_{\theta}$ = Value of a given variable with the grid specified in the subscript

Figure 12: $\gamma$ VERSUS $\eta$
\( S(\lambda) \) = Wall shear stress correlation

\( S^*, S^w \) = Entropy generation rate per unit surface area and pointwise entropy generation rate, respectively

\( T \) = Temperature

\( U, u \) = Mean velocity, and velocity component in the streamwise direction, respectively

\( U_G \) = Grid uncertainty

\( u_t \) = Friction velocity, \( \sqrt{\frac{\tau_w}{\rho}} \)

\( u', v', w' \) = Velocity fluctuations in the streamwise, wall-normal, and spanwise directions, respectively

\( \overline{u'v'} \) = Mean fluctuation produce in Reynolds shear stress

\( V_{avg}, V \) = Average velocity magnitude and velocity vector, respectively

\( V, v \) = Mean velocity, and velocity component in the plate normal direction, respectively

\( x, y \) = Coordinates in streamwise and wall-normal direction, respectively

Non-dimensional quantities

\( C_d \) = Dissipation coefficient, \( \frac{TS^*}{\rho U_{fs}^2} \)

\( C_f \) = Skin friction coefficient, \( \frac{\tau_w}{\rho U_{fs}^2} \)

\( Re \) = Reynolds number, the corresponding subscript is the characteristic length for the equation, \( Re_\theta = \frac{U_{fs} \theta}{\nu} \)

\( (S^*)^+ \) = Entropy generation rate per unit surface area, \( \frac{TS^*}{\rho u_t^2} \)

\( (S^w)^+ \) = Pointwise volumetric entropy generation rate, \( \frac{TVw^w}{\rho u_t^2} \)

\( U' \) = Mean velocity, \( \frac{U}{u_t} \)

\( y' \) = Wall-normal coordinate, \( \frac{y_w}{\nu} \)

Greek Symbols

\( \beta \) = Streamwise pressure gradient parameter

\( \beta_D \) = Hartree profile dissipation integral parameter, \( \int_0^\infty f^{\beta_D} \, d\eta \)

\( \Delta \) = Change in grid points along an axis

\( \delta, \delta' \) = Boundary layer thickness and displacement thickness, respectively

\( \delta_s \) = Percentage difference

\( \delta_{RE} \) = Error estimate

\( \varepsilon \) = Turbulent dissipation rate, \( m^2/s^3 \)

\( \varepsilon_{ex} \) = Changes between \( \hat{S}_n \) for different grids

\( \eta \) = Falkner-Skan parameter, \( \sqrt[3]{\frac{U_{fs}(\omega + m)}{2v_x}} \)

\( \eta_t \) = Transition length

\( \gamma_t \) = Intermittency based on \( C_f \)

\( \lambda \) = Thwaites' correlation parameter, \( \frac{\theta U_{fs}^2(x)}{v} \)

\( \mu \) = Absolute viscosity

\( \omega \) = Specific dissipation rate, \( 1/s \)

\( \theta \) = Momentum thickness

\( \nu \) = Kinematic viscosity, \( \frac{\mu}{\rho} \)

\( \nu_t \) = Turbulent viscosity

\( \rho \) = Density

\( \tau \) = Stress tensor

\( \tau_w \) = Wall shear stress

Superscripts

\((\_)^+\) = Normalization by wall units

Subscripts

\( 1, 2, 3 \) = Represent fine, medium and coarse grid, respectively

\( avg \) = Average of the value

\( D_h \) = Hydraulic diameter

\( fs \) = Freestream value

\( G \) = Grid convergence value

\( L \) = Length of the plate

\( 0 \) = Value at inlet

\( s \) = Static pressure

\( t \) = Total pressure

REFERENCES


